VECTOR CALCULUS

Field-

1. **Scalar Field.** $f(x,y,z) = x^2yz$

2. **Vector field.** $A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ **Where A_1, A_2, A_3 are scalar in terms of x, y and z**

Differentiation of Vectors-

Let R(u) be a vector which depends on u and is the variation in u. so we will find the variation in R.

$$\Rightarrow \frac{\Delta R}{\Delta u} = \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

$$lt_{\Delta u \to 0} \frac{\Delta R}{\Delta u} = \frac{dR}{du} = lt_{\Delta u \to 0} \frac{\Delta R(u+\Delta u) - R(u)}{\Delta u}$$

SPACE CURUE

$$= X(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$$

$$\frac{dR}{du} = \left(\frac{dx}{du}\right)\hat{i} + \left(\frac{dy}{du}\right)\hat{j} + \left(\frac{dz}{du}\right)\hat{k}$$

→ This is the differentiation wrt u if u is the function of 't' then du/dt represents the velocity of the particle on the curve

→ $\frac{d^2r}{dt^2}$ represents acceleration of the particle

CONTINUITY OF THE SCALAR

Let $'f(u)'$ be the scalar function which is continuous at u.

If,

$$Lt_{\Delta u \to 0} f(u+\Delta u) = f(u)$$

i.e. $\forall\ \varepsilon > 0$ and $\delta > 0$ such that

$|f(u+\Delta u) - f(u)| < \varepsilon$ whenever

$|u - \Delta u| < \delta$

Then phi is said to be continuous at u=0

PARTIAL DERIVATIVE OF THE SCALAR

Let A = A(x,y,z) be the scalar field then

$$\frac{\partial A}{\partial x} = Lt_{h \to 0} \frac{A(x+h,y,z) - A(x,y,z)}{h}$$

$$\frac{\partial A}{\partial y} = Lt_{K \to 0} \frac{A(x,y+K,z) - A(x,y,z)}{K}$$

and

$$\frac{\partial A}{\partial z} = Lt_{l \to 0} \frac{A(x,y,z+l) - A(x,y,z)}{l}$$

→ If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ where A_1, A_2, A_3 are continuous scalar fields then we can apply the same formulas for $\vec{A}$

PROPERTIES

(1) $$\frac{d}{du}\left(\vec{a} + \vec{b}\right) = \frac{d\vec{a}}{du} + \frac{d\vec{b}}{du}$$

(2) $$\frac{d}{du}\left(\vec{a} \cdot \vec{b}\right) = \frac{d\vec{a}}{du} \cdot \vec{b} + \vec{a} \cdot \frac{d\vec{b}}{du}$$

(3) $$\frac{d}{du}\left(\vec{a} \times \vec{b}\right) = \frac{d\vec{a}}{du} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{du}$$

(4) $\frac{d}{du}\left(f\ \vec{a}\right)=\frac{df}{du}\times\vec{a}+f\frac{d\vec{a}}{du}$

Similar properties are followed by partial derivatives (including higher order partial derivatives)

Note:- $\frac{\partial^2 t}{\partial x\partial y}=\frac{\partial^2 t}{\partial y\partial x}$ if f is continuous

Examples

(i) $\vec{r}=(\operatorname{Sin}t)\hat{i}+(\operatorname{Cos}t)\hat{j}+t\hat{k}$, find

(i) $\frac{d\vec{r}}{dt}$ (ii) $\frac{d^2\vec{r}}{dt^2}$ (iii) $\left|\frac{d\vec{r}}{dt}\right|$ (iv) $\left|\frac{d^2\vec{r}}{dt^2}\right|$

SOLU.-

$$\frac{d\vec{r}}{dt}=(\cos t)\hat{i}-(\sin t)\hat{j}+\hat{k}$$

$$\frac{d^2\vec{r}}{d^2t}=(-\sin t)\hat{i}-(\cos t)\hat{j}+0\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right|=\sqrt{1+1}=\sqrt{2}$$

$$\left|\frac{d^2\vec{r}}{dt^2}\right|=1$$

2. A particle moves along a curve whose parametric equations are $x=e^{-t}$, y = 2cos3t, z=2sin3t, find (i) velocity (ii) acceleration at any time t

SOLU.

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\vec{r}=(e^{-t})\hat{i}+(2\operatorname{Cos}3t)\hat{j}+(2\sin 3t)\hat{k}$$

$$\frac{d\vec{r}}{dt}=\text{velocity}=-e^{-t}\hat{i}-6\sin 3t\hat{j}+6\operatorname{Cos}3t\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2}=-e^{-t}\hat{i}-18\operatorname{Cos}3t\hat{j}-18\operatorname{Sin}3t\hat{k}$$

3. Find magnitude of velocity and acceleration in (2) at t=0

$$(\vec{v})=-\hat{i}-0\hat{j}+6\hat{k}$$

$$\Rightarrow\quad |\vec{v}|=\sqrt{1+36}=\sqrt{37}$$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{t=0}=a=\hat{i}-18\hat{j}$$

$$|a|=\sqrt{1+18^2}=\sqrt{325}$$

4. $\vec{A}=5t^2\hat{i}-t\hat{j}-t^3\hat{k}$

$\vec{B}=(\operatorname{Sin}t)\hat{i}-(\operatorname{Cos}t)\hat{j}$, find

$\frac{d}{dt}(\vec{A}\cdot\vec{B})$, $\frac{d}{dt}(\vec{A}\times\vec{B})$

SOLN.

$$\vec{A}\cdot\vec{B}=5t^2\operatorname{Sin}t+t\operatorname{Cos}t\hat{j}$$

And

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & -t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{i}(-t^3 \cos t) - \hat{j}(t^3 \sin t) + \hat{k}(t \sin t - 5t^2 \cos t)$$

$$\Rightarrow \quad \frac{d}{dt}(\vec{A} \times \vec{B}) = 10t \sin t + 5t^2 \cos t - t \sin t + \cos t$$

$$= 9t \sin t + 5t^2 \cos t + \cos t$$

$$\Rightarrow \quad \frac{d}{dt}(\vec{A} \times \vec{B}) = -\hat{i}(3t^2 \cos t + t^3 \sin t) - \hat{j}(3t^2 \sin t + t^3 \cos t) + \hat{k}(t \cos t + \sin t - 10t \cos t + 5t^2 \sin t)$$

$$= \quad \hat{i}(t^3 \sin t - 3t^2 \cos t) - \hat{j}(t^3 \cos t + 3t^2 \sin t) + \hat{k}(5t^2 \sin t - 9t \cos t + \sin t)$$

5. $\vec{A} = (\sin u)\hat{i} + (\cos u)\hat{j} + u\hat{k}$

$\vec{B} = (\cos u)\hat{i} - (\sin u)\hat{j} - 3\hat{k}$

$\vec{C} = 2\hat{i} + 3\hat{j} - \hat{k}$

Find $\frac{d}{du}\vec{A} \times (\vec{B} \times \vec{C})$ at u=0

SOLU.

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u & -\sin u & -3 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= \hat{i}(9 + \sin u) - \hat{j}(6 - \cos u) + \hat{k}(3\cos + 2\sin u)$$

$$\vec{A} \times \vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin u & \cos u & u \\ 9 + \sin u & \cos u - 6 & 3\cos u + 2\sin u \end{vmatrix}$$

$$= \hat{i}(3\cos^2 u + \sin^2 u - u\cos u + 6u) - \hat{j}(3\sin u \cos u + 2\sin^2 u - u \sin u - 9u) + \hat{k}(-9\cos u - 6\sin u)$$

$$\frac{d}{du}\vec{A} \times (\vec{B} \times \vec{C}) = \hat{i}(-3\sin 2u + 2\cos 2u - \cos u + u\sin u + 6) + \hat{k}((9\sin u - 6\cos u)$$

$$- \hat{j}(3\cos^2 u - 3\sin^2 u - 2\sin 2u - \sin u - u\cos u - 9)$$

At u=0

$$\frac{d}{du}(\vec{A} \times (\vec{B} \times \vec{C})) = 7\hat{i} + 6\hat{j} - 6\hat{k}$$

6. $x = 4t - \frac{t^2}{2}, \quad y = 3 + 6t - \frac{t^3}{6} \quad$ at $t = 2$

Find $\frac{d\vec{r}}{dt}$

SOLU. $\vec{r} = x\hat{i} + y\hat{j}$

$$\frac{d\vec{r}}{dt} = (u - t)\hat{i} + \hat{j}\left(6 - \frac{t^2}{2}\right)$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=2} = 2\hat{i} + 4\hat{j}$$

OPERATOR

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad \text{OR} \quad \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Operator-

1. Gradient (Scalar $\rightarrow$ Vector)
2. Divergence (Vector $\rightarrow$ Scalar)
3. Curl (Vector $\rightarrow$ Vector)

$\Rightarrow$ **Rotation of the Axis –**

$\Rightarrow$ **Translation of the Axis –**

$\Rightarrow$ **Rotation and Translation –**

INVARIANCE-

No change in the quantity under rotation or translation is called invariance

Let T be the temperature at point P(x,y,z) in frame of reference xyz and T' be temperature at P(x', y', z')in frame of reference x' y' z'

$$T(x,y,z) = T(x',y',z')$$
$$x' = l_{11}x + m_{12}y + n_{13}z$$
$$y' = l_{21}x + l_{22}y + n_{23}z$$
$$z' = l_{31}x + m_{32}y + n_{33}z$$

Note:

1. Gradient, divergence and curve is invariant under rotation, translation and rotation with translation
2. Laplacian operator is also invariant under rotation, translation and rotation with translation.

GRADIENT

Let f be the differentiable scalar function then grad is defined as,

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

(eg) find gradient of $f = xy^2z^3$ at (1, 2, -1).

SOLU. $\nabla f = \hat{i}(y^2z^3) + \hat{j}(2xyz^3) + \hat{k}(3xy^2z^2)$

$$(\nabla f)_{(1,2,-1)} = \hat{i}(4.(-1)) + \hat{j}(-4) + \hat{k}(3.4)$$

$$= -4\hat{i} - 4\hat{j} + 12\hat{k}$$

DIRECTIONAL DERIVATIVE OF f ALONG $\hat{a}$ -

Let $\hat{a}$ be a unit vector and f be any differentiable scalar field then directional derivative of f along $\hat{a}$ is defined as

$$[\nabla f \cdot \hat{a} = \hat{a}.\nabla f]$$

Gradient is Normal to Level Surface (Physical Significance of ∇)

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \qquad = \qquad \partial x\hat{i} + \partial y\hat{j} + \partial z\hat{k}$$

As $Q \rightarrow P,\ \partial x, \partial y, \partial z \rightarrow 0$

$\Rightarrow \delta x, \delta y, \delta z \rightarrow dx, dy, dz$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\Rightarrow \nabla f \cdot d\vec{r}$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$= df \qquad (\because f(x,y,z) = c)$

$= 0$

$\Rightarrow$ It proves that tangent at level surface is perpendicular ∇f

(eg) $f(x,y,z) = 3x^2y - y^3z^2$, find grad f at (1,2,-1)

SOLU.

$$\nabla f = \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z^2)$$

$$(\nabla f)_{(1,-2,-1)} = -12\hat{i} + \hat{j}(3 - 12) + \hat{k}(-16)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

(eg) $f = \log|r|$ find ∇f

SOLU. $\nabla f = \sum \hat{i}\dfrac{\partial}{\partial x}(\log r)$

$$\sum \hat{i}\left(\frac{1}{r}\cdot\frac{\partial r}{\partial x}\right) = \sum \hat{i}\left(\frac{1}{r}\cdot\frac{x}{r}\right)$$

$$\frac{\sum \hat{i}x}{r^2} = \frac{\vec{r}}{r^2}$$

(eg) find $\nabla\left(\dfrac{1}{r}\right)$

SOLU.

$$\sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \sum \hat{i}\left[\frac{-1}{r^2}\cdot\frac{x}{r}\right]$$

$$= \frac{-\vec{r}}{r^3}$$

(eg) find ∇r^n

SOLU. $\sum \hat{i}\left(nr^{n-1}\dfrac{x}{r}\right) = nr^{n-2}\cdot\vec{r}$

(eg) let R be the distance from the fixed point A(a,b,c) to any point P(x,y,z) find grad R

SOLU. $R = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

$$\nabla R = \frac{(x-a)\hat{i}}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} + \frac{(y-b)\hat{j}}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} + \frac{(z-c)\hat{k}}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

$$\nabla R = \frac{\vec{R}}{R}$$

MAXIMUM DIRECTIONAL DERVIATIVE-

Maximum directional derivative from point P(x,y,z) is modulus of grad at (z,y,z)

$\Rightarrow$ M.D.D $= \left[(\nabla f)_{x,y,z}\right]$

(eg) find directional derivatives of $f = x^2yz + yxz^2$ at (1,-2,-1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$

SOLU. $\nabla f = \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$

$$(\nabla f)_P = 8\hat{i} - \hat{j} - 10\hat{k}$$

$$\hat{a} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

$$\left[(\nabla\phi)_P\cdot\hat a = \frac{16+1+20}{3} = 37/3\right]$$

(eg) In what direction from P(2,1,-1) is the directional derivative of $\phi = x^2yz^3$ is maximum?

SOLU. DD is maximum in direction of $\nabla\phi$

$$\nabla\phi = \hat i(2xyz^3)+\hat j(x^2z^3)+\hat k(3x^2yz^2)$$

$$(\nabla\phi)_P = -4\hat i - 4\hat j + 12\hat k$$

$$\left|(\nabla\phi)_P\right| = \sqrt{16+16+144}$$

$$\sqrt{144+32} = \sqrt{176}$$

ANGLE B/W TWO SURFACES –

$$\cos\theta = \frac{(\nabla\phi_1)_P\cdot(\nabla\phi_2)_P}{\left|(\nabla\phi_1)_P\right|\left|(\nabla\phi_2)_P\right|}$$

(eg) find the angle b/w the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at (2,-1,2)

SOLU. Let $\phi_1 = x^2+y^2+z^2-9$ and ,

$$\phi_2 = -z + x^2+y^2-3$$

$\Rightarrow$ $(\nabla\phi_1) = \hat i(2x)+\hat j(2y)+\hat k(2z)$

$$(\nabla\phi_1)_P = 4\hat i - 2\hat j + 4\hat k$$

$$\left|(\nabla\phi_1)_P\right| = \sqrt{16+4+1} = \sqrt{21}$$

$\Rightarrow$ $(\nabla\phi_2) = \hat i(2x)+\hat j(2y)-\hat k$

$$(\nabla\phi_2)_P = 4\hat i - 2\hat j - \hat k$$

$$\left|(\nabla\phi_2)_P\right| = \sqrt{16+4+1} = \sqrt{21}$$

$\Rightarrow$ $\cos\theta = \dfrac{16+4-4}{6.\sqrt{21}} = \dfrac{16}{6.\sqrt{21}} = \dfrac{8}{3.\sqrt{21}}$

$$\theta = \cos^{-1}\left(\frac{8}{3.\sqrt{21}}\right)$$

Equation for Tangent Plane at Given Sore -

$$[(r-r_0)\cdot N = 0]$$

Or

$$\left[(\vec r - \vec r_0)\cdot\nabla\phi = 0\right]$$

DIVERGENCE –

Source	**Sink**	**Incompressible/Solnoidal**
$(\nabla\cdot u > 0)$	$(\nabla\cdot u < 0)$	$(\nabla\cdot u = 0)$

Let v be the differentiable vector field st. $\vec v = v_1\hat i + v_2\hat j + v_3\hat k$ where v_1, v_2, v_3 are differentiable scalar fields.

$$\nabla\cdot\vec v = \left(\hat i\frac{\partial}{\partial x}+\hat j\frac{\partial}{\partial y}+\hat k\frac{\partial}{\partial z}\right)\cdot(v_1\hat i + v_2\hat j + v_3\hat k)$$

$$= \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

(eg) $\vec A = x^2z\hat i - 2y^3z^3\hat j + xy^2z\hat k$ find div. $\vec A$ at (1,-1,1)

SOLU. $\nabla\cdot\vec{A} = 2xz - 6y^2z^3 + xy^2$

$$\left(\nabla\cdot\vec{A}\right)_{(1,-1,1)} = 2 - 6 + 1$$

$$= -3$$

Properties of Gradient-

(1) $\nabla(F+G) = \nabla F + \nabla G$

Proof $\nabla(F+G)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(F+G)$$

$$= \hat{i}\frac{\partial}{\partial x}(F+G) + \hat{j}\frac{\partial}{\partial y}(F+G) + \hat{k}\frac{\partial}{\partial z}(F+G)$$

$$= \hat{i}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial x}\right) + \hat{j}\left(\frac{\partial F}{\partial y} + \frac{\partial G}{\partial y}\right) + \hat{k}\left(\frac{\partial F}{\partial z} + \frac{\partial G}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}\right) + \left(\hat{i}\frac{\partial G}{\partial x} + \hat{j}\frac{\partial G}{\partial y} + \hat{k}\frac{\partial G}{\partial z}\right)$$

$$= \nabla F + \nabla G$$

(2) $\nabla(FG) = F\nabla G + G\nabla F$

Proof. $\nabla(FG)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(FG)$$

$$= \hat{i}\frac{\partial}{\partial x}(FG) + \hat{j}\frac{\partial}{\partial y}(FG) + \hat{k}\frac{\partial}{\partial z}(FG)$$

$$= \hat{i}\left(\frac{\partial F}{\partial x}G + F\frac{\partial G}{\partial x}\right) + \hat{j}\left(F\frac{\partial G}{\partial y} + G\frac{\partial F}{\partial y}\right) + \hat{k}\left(G\frac{\partial F}{\partial z} + F\frac{\partial G}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}\right)G + F\left(\hat{i}\frac{\partial G}{\partial x} + \hat{j}\frac{\partial G}{\partial y} + \hat{k}\frac{\partial G}{\partial z}\right)$$

$$= F\nabla G + G\nabla F$$

Laplacian Operator-

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \nabla\cdot\nabla f$$

Laplacian Equation –

$\nabla^2 f = 0$ $\quad$ f is called harmonic

(eg) $\nabla^2\left(\frac{1}{r}\right)$

SOLU. $\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\nabla\left(\frac{1}{r}\right)$

$$\nabla\left(\frac{1}{r}\right) = \sum \hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right) = -\frac{\vec{r}}{r^3}$$

$$\nabla\cdot\nabla\left(\frac{1}{r}\right) = -\sum \hat{i}\left[\frac{\hat{i}}{r^2} - \frac{3}{r^4}\vec{r}\,\frac{x}{r}\right]$$

$$= -\frac{3}{r^3} + 3\frac{r^2}{r^5} = 0$$

Properties-

(1) $\operatorname{div}(\vec{A} + \vec{B}) = \operatorname{div}\vec{A} + \operatorname{div}\vec{B}$

(2) $\operatorname{div}(f\vec{A}) = (\nabla f)\cdot\vec{A} + (\nabla f)\cdot\vec{B}$

Proof-

(1) $\operatorname{div}(\vec{A} + \vec{B}) = \nabla\cdot(\vec{A} + \vec{B})$

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\vec{A} + \vec{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

$$\nabla\cdot\left(\vec{A} + \vec{B}\right) = \left(\frac{\hat{i}\partial}{\partial x} + \frac{\hat{j}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)\cdot\left\{(A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}\right\}$$

$$= \left[\frac{\partial}{\partial x}(A_1 + B_1)\right] + \left[\frac{\partial}{\partial y}(A_2 + B_2)\right] + \left[\frac{\partial}{\partial z}(A_3 + B_3)\right]$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$= \left(\frac{\partial\hat{i}}{\partial x} + \frac{\partial\hat{j}}{\partial y} + \frac{\partial\hat{k}}{\partial z}\right)\cdot(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + \left(\frac{\partial\hat{i}}{\partial x} + \frac{\partial\hat{j}}{\partial y} + \frac{\partial\hat{k}}{\partial z}\right)\cdot(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$$

(2) Similar Proof

Physical significance of Divergence-

Let P represents the position of a particle during its venation in a field with velocity let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ where v_1 is velocity in the direction of x-axis and v_2 and v_3 represents velocities of particle in direction y and z axes respectively.

Þ Now, component of velocity at centre about phase ABCD

$$= \left(v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}dx\right)dydz$$

Þ Volume in the x-direction about EFGH

$$= \left(v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}dx\right)dydz$$

Þ So, total gain per unit time per unit volume is given by

$$= \left[\left(v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}dx\right) - \left(v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}dx\right)\right]dydz$$

$$= \frac{\partial v_1}{\partial x}\cdot dxdydz$$

$\Rightarrow$ similarly gain along y and z axes are $\frac{\partial v_2}{\partial y}\cdot dx\,dy\,dz$ and $\frac{\partial v_3}{\partial z}\cdot dx\,dy\,dz$

$\Rightarrow$ So, total gain per nit time per unit volume will be,

$$\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)dx\,dy\,dz$$

$\Rightarrow$ So, total gain per unit time will be,

$$\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

(At the cube shrinks to point P. So, $dx, dy, dz \to 0$

$$\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

$$= \nabla\cdot\vec{v}$$

Note:

1. If $\nabla\cdot\vec{v} > 0$, point P works as a source. The fluid flows outside the fluid is EXPANDIBLE. In this case, density decreases at P.
2. If $\nabla\cdot\vec{v} < 0$, point P works as a sink. The fluid is COMPRESSIBLE. (density increases at P).
3. If $\nabla\cdot\vec{v} = 0$, P works neither as source nor as sink. This equation is called CONTINUITY EQUATION OF INCOMPRESSIBLE FLUID. And such $\vec{v}$ is called solenoidal vector.

CURL

Let $\vec{v}$ be the differentiable vector field then curl $\vec{v}$ is defined as

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Where $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$\Rightarrow$ **NOTE:**

$$\boldsymbol{\nabla\times\nabla\phi = 0}$$

PROOF: $\nabla\times\nabla\phi = \nabla\times\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial x & \partial\phi/\partial x \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)-\hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial z\partial x}\right)+\hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= 0$$

If for any vector field $\vec{F}$ curl $\vec{F} = 0$ then we will always find a scalar field ϕ such that $\vec{F} = \nabla\phi$ and such ϕ is called SCALAR POTENTIAL.

PHYSICAL SINGNIFICANCE OF CURL –

At point P, we find the curl $\vec{F}$.

1. **Curl $\vec{F} = 0$ –** No rotation at point P such a field is called IRROTATIONAL FIELD.

2. **Curl $\vec{F} \neq 0$ -** At point P, the liquid rotates. If in any magnetic field $\vec{F}$ inside the capacitor, curl $\vec{F} = 0$ and outside the capacitor curl $\vec{F} \neq 0$ the current density increases and current works as d.c.

PROPERTIES

1. Curl Curl $(\vec{A} + \vec{B}) = \text{Curl}\,\vec{A} + \text{Curl}\,\vec{B}$

LHS. $= \nabla \times (\vec{A} + \vec{B})$

$$= \Sigma\, \hat{i} \frac{\partial}{\partial x} \times (\vec{A} + \vec{B}) \quad = \quad \Sigma\, \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \Sigma\, \hat{i} \times \frac{\partial \vec{A}}{\partial x} + \Sigma\, \hat{i} \times \frac{\partial \vec{B}}{\partial x}$$

$$= \nabla \times \vec{A} + \nabla \times \vec{B} = \text{RHS}$$

2. $\nabla \cdot (\nabla \times \vec{v}) = 0$

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_3}{\partial y \partial x} - \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_1}{\partial y \partial z}$$

$= 0$

3. $\nabla \cdot (\vec{a} \times \vec{b}) = (\nabla \times \vec{a}) \cdot \vec{b} - (\nabla \times \vec{b}) \cdot \vec{a}$

DIRECTIONAL DERIVATIVE OF A SCAALAR FUNCTION :

Along any line whose de's are l, m, n

Let f be the differentiable scalar function then directional derivatives is given by,

$$\frac{df}{dr} = Lt_{PQ \to 0} \frac{f(Q) - f(P)}{PQ} \text{ exist}$$

$$= Lt_{r \to 0} \frac{f(x',y',z') - f(x,y,z)}{r} \text{.......................(1)}$$

Equation of line PQ is given by,

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} = r$$

$\Rightarrow$ x' = x+ lr, y' = y + mr, z' = z + nr

$$\Rightarrow \quad \frac{df}{dr} = Lt_{r \to 0} \frac{f(x + lr, y + mr, z + nr) - f(x,y,z)}{r}$$

Using Taylor's expansion,

$$\frac{df}{dr} = Lt_{r \to 0} \frac{f(x,y,z) + lr\frac{\partial f}{\partial x} + mr\frac{\partial f}{\partial y} + nr\frac{\partial f}{\partial z} - f(x,y,z)}{r}$$

$$= Lt_{r \to 0} \quad l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}$$

$$= \frac{\partial \phi}{\partial x} l + \frac{\partial \phi}{\partial y} m + \frac{\partial \phi}{\partial z} n$$

$$= \nabla \phi \cdot (l\hat{i} + m\hat{j} + n\hat{k})$$

$$= \frac{\nabla \phi \cdot (a\hat{i} + b\hat{j} + c\hat{k})}{\sqrt{a^2 + b^2 + c^2}} \quad \text{(a,b,c,d are dr's)}$$

$$= \nabla \phi \cdot \hat{a} \text{........................(2)}$$

NOTE:

1. D.D along x-axis- We know that dc's x-axis are so from (2)

$$\phi \quad D.D = \frac{\partial \phi}{\partial x}$$

2. D.D along y-axis- is $\frac{\partial \phi}{\partial y}$ and along z-axis is $\frac{\partial \phi}{\partial z}$

$\Rightarrow$ Snow that the maximum directional derivatives of ϕ (i.e) the maxi. rate of change of ϕ) along the direction of $\nabla \phi$.

Let $\hat{n}$ be a unit vector normal to surface ϕ . We know that $\nabla \phi$ is also normal to the surface so $\hat{n}$ will be parallel to $\nabla \phi$

$\Rightarrow$ $\nabla \phi = A\hat{n}$......................(1)

We have D.D along unit vector $\hat{n}$

$$\frac{d\phi}{dn} = \nabla \phi \cdot \hat{n}$$

$$= A\hat{n} \cdot \hat{n}$$

$$= A$$

$\Rightarrow$ $\nabla \phi = \frac{d\phi}{dn} \cdot \hat{n}$

$$= \left|\frac{d\phi}{dn}\right| |\hat{n}| \cos \theta$$

$$= \left|\frac{d\phi}{dn}\right|$$

This quantity will be maxi. when $\cos \theta$ is maximum (i.e. $\theta = 0$). Hence, maximum directional derivative is along $\nabla \phi$ and its value is $|\nabla \phi|$.

$\Rightarrow$ $\nabla \phi$ is invariant under rotation of rectangular axes.

We know that ,

$$r = (r.\hat{i})\hat{i}' + (r.\hat{j})\hat{j}' + (\vec{r}.\hat{k})\hat{k}'$$

$\hat{i}', \hat{j}', \hat{k}'$ are unit vectors wrt. x', y', z' axes respectively after rotation

Let ϕ be the scalar field and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x,y,z axis respectively so, we have

$$\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \text{........................(1)}$$

Rotate the axes about O and we find another frame of reference x', y', z' and let i', j', k' are unit vectors along x', y' and z' respectively.

So D.D along x' is given by $= \hat{i}'.\nabla \phi$

DD along y' is $= \hat{j}'.\nabla \phi$

DD along z' is $= \hat{k}'.\nabla \phi$

So $\text{grad}'\, f = \hat{i}'(\hat{i}'.\nabla f) + \hat{j}'(\hat{j}'.\nabla f) + \hat{k}'(\hat{k}'.\nabla f)$

$$= \hat{i}'\left[\hat{i}'\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\right] + \hat{j}'\left[\hat{j}'\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\right] + \hat{k}'\left[\hat{k}'\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\right]$$

$$= \sum \left\{\hat{i}'(\hat{i}'.\hat{i}) + \hat{j}'(\hat{j}'.\hat{j}) + \hat{k}'(\hat{k}'.\hat{k})\right\}\frac{\partial f}{\partial x}$$

$$= \sum \hat{i}\frac{\partial f}{\partial x}$$

$$= \left\{\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right\}$$

$$= \nabla f$$

$\Rightarrow \quad \text{grad} f = \text{grad}' f$

LINE INTEGRAL

$ds^2 = dx^2 + dy^2$

In 3-D,

$ds^2 = dx^2 + dy^2 + dz^2$

$ds = \sqrt{dx^2 + dy^2 + dz^2}$

$$\left[\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\right.$$ v(t) is velocity of particle

$$\left.\frac{dS}{dt} = |v(t)|\right]$$

LINE INTEGRAL

1. **over the function**
2. **over the vector field.**

$\Rightarrow$ Let f(x,y,z) be a continuous function and c be any smooth curve. Let a particle moves along the curve from A' to B' then find work done by the particle over the function on the curve c. Let curve c be divided in n-sub arcs and let one sub-arc is Δsi long.

$$S_n = \sum_{i=1}^{n} f(x,y,z)\Delta si \qquad (\text{as } n \to \infty, \Delta si \to 0)$$

$$\text{Let}_{n\to\infty} \; S_n = \int_c f(x,y,z)ds$$

$$I = \int_c f(x,y,z)ds \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

If c be any closed curve then $I = \oint F(x,y,z)ds$ is called CIRCULATION of the function over curve c.

$$I = \int_c f(x,y,z)\frac{ds}{dt}.dt$$

$\Rightarrow$ $$I = \int_c f(x,y,z) \; |v(t)| \; dt \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

$\Rightarrow$ If $r = g(t)\hat{i} + k(t)\hat{j} + h(t)\hat{k}$

$$I = \int_c f[g(t),k(t),h(t)] \; |v(t)| \; dt$$

$\Rightarrow$ If $r = x\hat{i} + y\hat{j} + z\hat{k}$

$$I = \int_c f(x(t), y(t), z(t)) \ |v(t)| \ dt$$

(e.g.) $I = \int_v (x + y)ds,$ c is straight line segment x = t, y = 1-t, z = 0 from (0,1,0) to (1,0,0)

SOLU. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = t\hat{i} + (1 - t)\hat{j}$

$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j}$

$|v(t)| = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{2}$

$\Rightarrow \int_c (x + y) \ ds = \int_c^1 (t + 1 - t).\sqrt{2} \ dt = \sqrt{2}$

(e.g.) $\int_c (x - y + z - z)ds$ where c is a straight line x = t, y = 1-t, z = 1 from (0,1,1) to (1,0,1)

SOLU. $\vec{r} = t\hat{i} + (1 - t)\hat{j} + \hat{k}$

$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j} \qquad |v(t)| = \sqrt{2}$

$\Rightarrow \int_c (x - y + z - z)ds = \int_0^1 (t - 1 + t + 1 - 2)\sqrt{2} \ dt$

$= \int_c^1 (2t - 2)\sqrt{2} \ dt$

$= 2\sqrt{2}\left[\frac{t^2}{2} - t\right]_0^1 = 2\sqrt{2}\left(\frac{1}{2} - 1\right)$

$= -\sqrt{2}$

(e.g.) $\int_c (xy + y + z)ds$ along the curve

$r = 2t\hat{i} + t\hat{j} + (2 - 2t)\hat{k}$ from t = 0 to 1.

SOLU. $\frac{d\vec{r}}{dt} = 2\hat{i} + \hat{j} - 2\hat{k}$

$|v(t)| = \sqrt{4 + 4 + 1} = 3$

$\int_c (xy + y + z)ds = \int_0^1 (2t^2 + t + 2 - 2t)(\sqrt{3}) \ dt$

$= 3\int_0^1 (2t^2 - t + 2) \ dt$

$$= 3\left[\frac{2t^3}{3} - \frac{t^2}{2} + 2t\right]_0^1$$

$$= 3\left[\frac{2}{3} - \frac{1}{2} + 2\right] = 3\left[\frac{2}{3} + \frac{3}{2}\right] = 13/2$$

(e.g) find the integration over $f(x,y)=x^3/y$ where c: $y = \frac{x^2}{2}$ and x lies b/w o to 2

SOLU. $\int_c f(x,y)ds = \int_c f(x,y)\frac{ds}{dx}\,dx$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad (y = x^2/2)$$

$$= \sqrt{1+x^2}$$

So, $$\int_c f(x,y)ds = \int_0^2 \frac{x^3}{y}\sqrt{1+x^2}\,dx$$

$$= \int_0^2 \frac{2x^3}{x^2}\sqrt{1+x^2}\,dx$$

$$= \int_0^2 2x\sqrt{1+x^2}\,dx$$

$$= \frac{2}{3}\left(5\sqrt{5} - 1\right)$$

(e.g.) $F(x,y) = x^2 - y$ c: $x^2 + y^2 = 4$ in 1st quad. From (0,2) to $(\sqrt{2},\sqrt{2})$

SOLU. $x = 2\cos\theta$ $y = 2\sin\theta$

$$\frac{ds}{dt} = 2$$

$$= \int_{\pi/2}^{\pi/4} (4\cos^2\theta - 2\sin\theta)2d\theta$$

$$= 2\int_{\pi/2}^{\pi/4} 2(1+\cos 2\theta)d\theta - 4\int_{\pi/2}^{\pi/4} \sin\theta\, d\theta$$

$$= -\left(\pi - 2(1+\sqrt{2})\right)$$

$$= 2(1+\sqrt{2}) - \pi$$

Line Integration over the Vector Field-

$\Rightarrow$ $$W = \int_c \vec{F}.\vec{T}ds$$

$$W = \int_c \vec{F}.d\vec{r}$$

$\Rightarrow$ If $\vec{r}$ be any general vector

$$\vec{r} = g(t)\hat{i} + h(t)\hat{j} + w(t)\hat{k}$$

$$a \le t \le b$$

$$d\vec{r} = g'(t)\hat{i} + h'(t)\hat{j} + w'(t)\hat{k}$$

$\Rightarrow$ $W = \int_a^b \vec{F}.\{g'(t)\hat{i} + h'(t)\hat{j} + w'(t)\hat{k}\}.dt$

$$\left[W = \int_a^b \vec{F}.\{Mg'(t) + Nh'(t) + Lw'(t)\}.dt\right] \text{.......................(1)}$$

$\Rightarrow$ If $\vec{r}$ be any position vector, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}$

$$\left[W = \int_c \vec{F}.d\vec{r} = \int_c Mdx + Ndy + Ldz\right] \text{...................(2)}$$

$\Rightarrow$ If $\vec{r}$ be any general vector $\vec{r} = g(x)\hat{i} + h(x)\hat{j} + w\hat{k}$

$d\vec{r} = (dg)\hat{i} + (dh)\hat{j} + (dw)\hat{k}$

$\Rightarrow$ $\int_c \vec{F}.d\vec{r} = Mdg + Ndh + Ldw$

$\Rightarrow$ If d is any closed curve then work done is called circulation

$$\left[W = \oint \vec{F}.d\vec{r}\right]$$

(e.g) Let $\vec{F} = x^2\hat{i} + y^2z\hat{j} + z^2\hat{k}$ be the vector field. Find work done along the curve x = cost, y = ysint.

$0 \le t \le \pi/2, \ z = t$

SOLU. $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$

$d\vec{r} = -\sin t\hat{i} + \cos t\hat{j} + k$

$\int_c \vec{F}.d\vec{r} = \int_0^{\pi/2} (-\cos^2 t\sin t + \sin^2 t\cos t + t^2)dt$

Path Independent

Any work done from A to B does not depend on the path followed and depends only A and B, such work done is called path independent and such field is called CONSERVATIVE FIELD.

Theorem- If $W = \int_c \vec{F}.d\vec{r}$ is path independent iff $\vec{F} = \nabla f$ (such f is called scalar potential)

Proof- Let $\vec{F} = \text{grad} f$

Now, we will prove that is path independent i.e. we will show that the integral depends only on initial and final points.

$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = \int_A^B df$

$= f(B) - f(A)$

Hence, it is path independent CONVERSELY,

Let $W = \int_C \vec{F}\cdot d\vec{r}$ is path independent

Let $f(x,y,z) = \int_{(x,y,z)}^{(x',y',z')} \vec{F}\cdot d\vec{r}$

$$f = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right) ds$$

$$\frac{df}{ds} = \vec{F} \cdot \frac{d\vec{r}}{ds} \quad ....(1)$$

We know that

$$\nabla f . d\vec{r} = df$$

$$\nabla f \cdot \frac{d\vec{r}}{ds} = \frac{df}{ds} \quad ....(2)$$

From (1) and (2)

$$\vec{F} \cdot \frac{d\vec{r}}{ds} = \nabla f . \frac{d\vec{r}}{ds}$$

$$(\vec{F} - \nabla f) \cdot \frac{d\vec{r}}{ds} = 0 \qquad \left(\frac{d\vec{r}}{ds} \neq 0, \text{tangent along as}\right)$$

$$\vec{F} - \nabla f = 0$$

$$\left[\vec{F} = \nabla f\right]$$

THEOREM(2)-

Let D is closed connected domain then $\oint_C \vec{F}.d\vec{r}$ is path independent iff $\oint_C \vec{F}.d\vec{r} = 0$

Proof: Let $W = \oint_C \vec{F}.d\vec{r}$ is path independent i.e. $\oint_C \vec{F}.d\vec{r}$ depends only on A and B and not on path followed i.e.

$\exists\, f$ (scalar potential) such that

$$\vec{F} = \nabla f$$

$$\Rightarrow \quad W = \oint_C \vec{F}.d\vec{r} = \oint_C \nabla f . d\vec{r}$$

$$= \int_A^A df$$

$$= f(A) - f(A) = 0$$

Conversely,

Let $\oint_C \vec{F}.d\vec{r} = 0$ we will prove that $\oint_C \vec{F}.d\vec{r}$ is path independent

$$\Rightarrow \quad \oint_C \vec{F}.d\vec{r} = 0$$

$$\oint_{ACBDA} \vec{F}.d\vec{r} = 0$$

$$\int_{ACB} \vec{F}.d\vec{r} + \int_{BDA} \vec{F}.d\vec{r} = 0$$

$$\int_{ACB} \vec{F}.d\vec{r} - \int_{ADB} \vec{F}.d\vec{r} = 0$$

$$\int_{ACB} \vec{F}.d\vec{r} = \int_{ADB} \vec{F}.d\vec{r}$$

$$\left[\int_{P_1} \vec{F}.d\vec{r} = \int_{P_2} \vec{F}.d\vec{r}\right]$$

Which proves that work done is independent of path followed.

THEOREM (3):

$\int_c \vec{F}\cdot d\vec{r}$ is path independent for any c (either closed or open) iff curl $\vec{F} = 0$

Proof: if $W = \int \vec{F}\cdot d\vec{r}$ is path independent, and f st.

$\vec{F} = \nabla f$

$\Rightarrow \quad \nabla \times \vec{F} = \nabla \times (\nabla f) = 0$

Conversely,

If $\nabla \times \vec{F} = 0$ and f st

$\vec{F} = \nabla f$

So by theorem (1)

$\int_c \vec{F}\cdot d\vec{r}$ is path independent

Exact Differential form:-

Let $\vec{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ be differentialable vector field. It is called differential (Mdx+Ndy+Pdz) form if s.t.

$$F = \frac{\partial f dx}{\partial x} + \frac{\partial f dy}{\partial y} + \frac{\partial f dz}{\partial z}$$

$\Rightarrow \quad \left[\vec{F} = df\right]$

$\Rightarrow$ If $\vec{F}$ has exact differential equation then

$\left[\nabla \times \vec{F} = 0\right]$ and such $\vec{F}$ is called conservative field.

Green Theorem on the Plane

Let $\int_c \vec{F}\cdot d\vec{r}$ is not path independent i.e. $\vec{F}$ is not associated by conservative field then green's theorem goes from line integral to double integral.

Mathematical Condition:

1. CONTINUTIY OF M and N $\left(\vec{F} = M\hat{i} + N\hat{j}\right)$-

   $M(x,y) = M$ and $N = N(x,y)$

   M ,N and their first order partial derivatives (i.e. M_x, M_y, N_x, N_y) are all continuous

2. Condition on C – C is simple closed curve piecewise smooth.

$\hat{k}$ - Component of Curl $\vec{F}$ -

Let $\vec{F} = M\hat{i} + N\hat{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix}$$

$$= \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\left(\nabla \times \vec{F}\right)\cdot\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$\Rightarrow \quad \nabla \cdot \vec{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

Outward Flux of B-

W= circulation or work done $= \int_C \vec{F}\cdot d\vec{r}$..........................(1)

$$= \int_C \vec{F}\cdot\left(\frac{d\vec{r}}{ds}\right)ds$$

$$= \int_C \vec{F}\cdot T\,ds$$

$$[T = K\times\hat{n}]$$

$$W = \int_C \vec{F}\cdot(k\times\hat{n})ds$$

$$W = \int_C (\vec{F}\times k)\cdot\hat{n}ds = \int_C B\cdot n\,ds$$

$$\int_C B\cdot\hat{n}\,ds = \int_C Mdy - Ndx$$

$\Rightarrow$ Outward flux of B

GREEN'S THEOREM

1. Out ward Flux of $\vec{F}$

$$\oint_C \vec{F}\cdot\hat{n}ds = \oint_C Mdy - Ndx$$

$$= \iint_R \left(\frac{\partial N}{\partial y} + \frac{\partial M}{\partial x}\right)dxdy$$

$$= \iint_R \left(\nabla\cdot\vec{F}\right)dxdy$$

2. Circulation of $\vec{F}$

$$\oint_C Mdx + Ndy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= \iint_R \left(\nabla\times\vec{F}\right)\cdot\hat{k}\,dxdy$$

Proof:- **always used for anticlockwise direction…**

$$= \int_a^b \left(\int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial M}{\partial y}dy\right)dx$$

$$= \int_a^b [M(x,y)]_{y=f_1(x)}^{y=f_2(x)}dx$$

$$= \int_a^b \{M(x,f_2) - M(x,f_1)\}dx$$

$$= -\int_a^b M(x,f_1) + \int_a^b M(x,f_2)dx$$

$$= -\int_b^a M(x,f_2) - \int_a^b M(x,f_1)\ dx$$

$$= -\left[\oint_{C_1} Mdx + \oint_{C_2} Mdx\right]$$

$$= -\oint_{c} Mdx$$

$$\text{So, } \oint_{c} Mdx = -\iint_R \left(\frac{\partial M}{\partial y} dy\right) dx \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{So, } \iint_R \frac{\partial N}{\partial x} dxdy = \oint_{c} Ndy \ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2)

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R \left(N_x - M_y\right) dxdy$$

(e.g.) $F = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$ $\qquad h^2 \le x^2 + y^2 \le 1$

Find $\oint_C \vec{F}.d\vec{r}$ $\qquad$ (i) $0 < h < 1$ $\quad$ (ii) $h = 0$

SOLU. $M = \frac{-y}{x^2+y^2}$

$N = \frac{x}{x^2+y^2}$

$\Rightarrow \quad \nabla \times \vec{F} = 0$

Here $M_y = N_x = \frac{y^2 - x^2}{x^2 + y^2}$

In this annulus region, origin is not included. So, M , N, My, Nx are all continuous. So, by applying green's theorem

$$\oint_C \vec{F}.d\vec{r} = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$= 0$

(ii) If $h = 0$

Here the function will be discontinuous at (0,0) as origin is induded within the region. Hence Green's theorem can't be applied. So we assume an elementary circle C1 excluding (0,0). So, by applying Green's theorem,

$$\oint_{\tau} + \oint_{C_1} = 0$$

$$\oint_{\tau} = -\oint_{C_1(\text{clockwise})} = \oint_{C_1(\text{anticlockwise})}$$

$$\oint_{C_1} Mdx + Ndy = \oint_{C_1} \frac{-ydx + xdy}{x^2 + y^2}$$

$$= \int_0^{2\pi} d\tan^{-1}(y/x)t$$

$$= \left[\tan^{-1}(\tan\theta)\right]_0^{2\pi}$$

$= (\theta)_0^{2\pi} = 2\pi$

..Surface Area and Surface Integral..

Let S be the curved surface and find the projection of curved surface in x-y plane but this projection is one-one and one with the curved surface then we can find integral along the surface which is called surface integral over s, surface integral has some physical importance in mechanism of membrane parashoot etc.

Surface area of s is given by ,

$$\left[S_a = \iint_R \frac{|\nabla f|}{|\nabla f . P|} dA \right]$$

Where P is unit normal of R (projection) and ∇f is normal to surface s.t. $\nabla f . P \neq 0$

(e.g.) find the area of the surface cut from bottom of paraboloid $x^2 + y - z = 0$ and $z = y$

S: $x^2 + y^2 - z = 0$

$x^2 + y^2 = y$ (Projection)

$z = y$

$$S_A = \iint \frac{|\nabla f|}{|\nabla f . P|} dA$$

$\Rightarrow \quad \nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k} \qquad P = \hat{k}$

$$|\nabla f| = \sqrt{1 + 4(x^2 + y^2)}$$

$$|\nabla f . P| = 1$$

$$S_A = \iint \sqrt{1 + 4(x^2 + y^2)}\, dxdy$$

$$S_A = \int_0^2 \int_0^{2\pi} \sqrt{1 + 4r^2}\, r dr d\theta$$

$$S_A = 2\pi \int_0^2 r\sqrt{1 + 4r^2}\, dr$$

$$S_A = \frac{2\pi}{8} \int_1^{17} \sqrt{t}\ dt$$

$$S_A = \frac{\pi}{6}(17\sqrt{17} - 1)$$

(e.g.) find the area of cap out by sphere $x^2 + y^2 + z^2 = 2,\ z \geq 0$ by the cylinder $x^2+y^2=1$

SOLU. R: $x^2 + y^2 = 1$

F: s: $x^2+y^2+z^2-2$

$P = \hat{k}$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}$$

$$|\nabla f| = 2\sqrt{2}$$

$$|\nabla f \cdot P| = 2z$$

$$S_A = \iint \frac{2\sqrt{2}}{2z} dxdy = \iint \frac{\sqrt{2}}{\sqrt{2 - x^2 - y^2}} dxdy$$

$$= \sqrt{2} \int_0^1 \int_0^{2\pi} \frac{r dr d\theta}{\sqrt{2 - r^2}}$$

$$= 2\sqrt{2}\pi\int_0^1 \frac{r\,dr}{\sqrt{2-r^2}}dr = -\sqrt{2}\pi\int_2^1 \frac{dt}{\sqrt{t}}$$

$$= 2\sqrt{2}\pi\left(\sqrt{2}-1\right)$$

SURFACE INTEGRAL-

Let S be the given surfaces such that F(x,y,z)=C and G (x,y,z) be continuous function in projection region r of S. then surface integral of G over F is given by

$$S = \iint G(x,y,z)\frac{|\nabla f|}{|\nabla f\cdot P|}\, dA$$

$$\text{Let } ds = \frac{|\nabla f|}{|\nabla f\cdot P|}\, dA$$

$$S = \iint_R G\,ds$$

ORIENTABLE SURFACE OR TINO SIDE SURFACE-

A surface S is called orientable surface if surface has two side or a surface is called Orientable if normal of one side is opposite of second side or whenever we will walk in the surface from one point and back to same point then direction of normal will be same.

FLUX OF THE SURFACE OR SURFACE INTEGRAL-

Let $\vec{F}$ be a continuous vector field and S is smooth orientable surface, $\hat{n}$ n be unit normal to surface S then flux of surface is given by

$$I = \iint_S \vec{F}\cdot\hat{n}\,ds$$

If a surface S is a level surface such that f(x,y,z)=C

$$\hat{n} = \pm\frac{\nabla f}{|\nabla f|} \qquad [+ve - \text{ anticlockwise}, \quad -ve - \text{ clockwise}]$$

$$I = \iint_S \left(\vec{F}\cdot\frac{\nabla f}{|\nabla f|}\right)\frac{\nabla f}{|\nabla f\cdot P|}dxdy$$

$$\left[I = \iint_S \left(\vec{F}\cdot\nabla f\right)\frac{dxdy}{|\nabla f\cdot P|} \right]$$

STROKE'S THEOREM

We know that Green's theorem is applicable only for plane region. Now, Stroke's theorem will be applied in the space region. So this theorem is also called as GENERATLISATION OF GREEN'S THEOREM.

STATEMENT- let S be smooth orientable surface & I is (+ve)ly oriented with respect to surface s then

$$\oint_\tau \vec{F}.d\vec{r} = \iint_S \left(\nabla\times\vec{F}\right).\hat{n}ds$$

$\Rightarrow$ For $\hat{n}$ apply right handed thumb

$$\begin{bmatrix} \hat{n} = +\frac{\nabla f}{|\nabla f|} \Rightarrow \text{ Positively oriented (anticlockwise)} \\ \hat{n} = -\frac{\nabla f}{|\nabla f|} \Rightarrow \text{ Negatively oriented (clockwise)} \end{bmatrix}$$

PARITCULAR CASES-

1. if we take $\hat{k}-$ component of unit vector $\hat{n}(0,0,1)$ then

$$\iint_S \left(\nabla\times\vec{F}\right).\hat{k}ds = \oint_\tau \vec{F}d\vec{r}$$

2. $f(b) - f(a) = \int_a^b f'(x)dx$

3. $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{k}\,ds$ (Plane)

4. $\oiint_S \vec{F}\cdot \hat{n}\,ds = \iiint_V (\nabla \cdot \vec{F})\,dxdydz$ (space)

5. $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\,ds$ (space)

$\Rightarrow$ For GREEN and STOKE'S boundary should be closed
But for GAUSS DIRVERGENCE, surface should be closed

$\Rightarrow$ **NOTE:**

1. If $F \in C' \Rightarrow F \notin C^2$ $(F = \nabla f)$
2. if $f \notin C^2 \Rightarrow F \notin C'$ $(F = \nabla f)$
3. If any function $G \in C^2$ then

$$\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x}$$

4. If $f_{xy} \neq f_{yx}$

$\Rightarrow$ $f \notin C^2$

Following statements are eqal-

Let f be continuous and curve C be piecewise smooth and surface S be a smooth curve.

i. F is conservative $\oint_\tau \vec{F}\cdot d\vec{r} = 0$

ii. If f is conservative, $W = \int_\pi^B \vec{F}\cdot d\vec{r}$ is independent of path

iii. $F = \nabla f$ $\left(\int_\pi^B \vec{F}\cdot d\vec{r} = f(B) - f(A)\right)$

If $F \in C'$ and S is open connected domain then $\nabla \times \vec{F} = 0$

(e.g.) $\int_C (\cos x \sin y - xy)dx + (\sin x \cos y)dy$

$x^2 + y^2 = 1$

$f_x = \cos x \sin y - xy$ $f_y = \sin x \cos y$

$f_{xy} = \cos x \cos y - x$ $f_{yx} = \cos x \cos y$

$f_{xy} \neq f_{yx}$

$\Rightarrow$ $f \notin C^2$

$\Rightarrow$ $F \notin C'$

$\int_C \vec{F}\cdot d\vec{r} = 0$ through $\nabla \times \vec{F} \neq 0$ (as $f \notin C^2$)

1. If $\vec{F}$ is continuous,
$\vec{F}$ is conservative $\Leftrightarrow$ $\vec{F} = \nabla f$

c c

$\oint_C \vec{F}\cdot d\vec{r} = 0 \Rightarrow \nabla\times\vec{F} = 0\ \forall D$

(Closed curve)

2. If $\vec{F}\in C'$

$\vec{F}$ is conservative $\Leftrightarrow \vec{F} = \nabla f$

$\oint_C \vec{F}\cdot d\vec{r} = 0 \Leftrightarrow \nabla\times\vec{F} = 0\ \forall D$

DIVERGENCE THEOREM

1. Surface should be closed.
2. $\vec{F}\in C'$

Statement- Let S be a closed surface of region V and $\vec{F}\in C'$ then

$$\left[\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dV\right]$$

$\hat{n}$ is outward normal to S.

This theorem is generalization of normal from of Green's theorem

$$\oint_C \vec{F}\cdot\hat{n}\,dS = \iint_S (\nabla\cdot\vec{F})\,dx\,dy$$

Green's Theorem-

1. Normal Green's Theorem

$$\oint_C \vec{F}\cdot\hat{n}\,dS = \iint_S (\nabla\cdot\vec{F})\,dx\,dy$$

$\overline{3-D}\,(\vec{F}\in C')$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_S (\nabla\cdot\vec{F})\,dx\,dy\,dz$$

2. Tangential Green's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{k}\,ds$$

$\overline{3-D}$

$$= \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

(e.g) Apply divergence theorem for,

$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$\nabla\cdot\vec{F} = 3$

$= \iint_S \vec{F}\cdot\hat{n}\,dS = \iiint (\nabla\cdot\vec{F})\,dV$

$= 3\int_V dV$

$= 3\times \pi r^2 h = 3\times\pi\times(1) = 3\pi$

How to Find Unit of Volume Integral –

$V = \iiint f(x,y,z)\,dV$

$V = \iiint dV$ if $f(x,y,z) = 1$

⇒ For limit of z –

Take projection in X-Y plane and draw a time II to z-axis passing through typical point.

For limit of X

Take projection in Y-Z plane and draw a line II to X-axis passing through typical point.

$$V = \int_{a}^{b}\int_{f_1(x)}^{f_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dx\,dy\,dz$$

(e.g) Find volume of intersection of $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

$$x^2 + 3y^2 = 8 - x^2 - y^2$$

$$x^2 + 2y^2 = 4$$

$$V = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dx\,dy\,dz$$

$$V = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left(8 - x^2 - y^2 - x^2 - 3y^2\right)dxdy$$

$$V = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left(8 - 2x^2 - 4y^2\right)dxdy$$